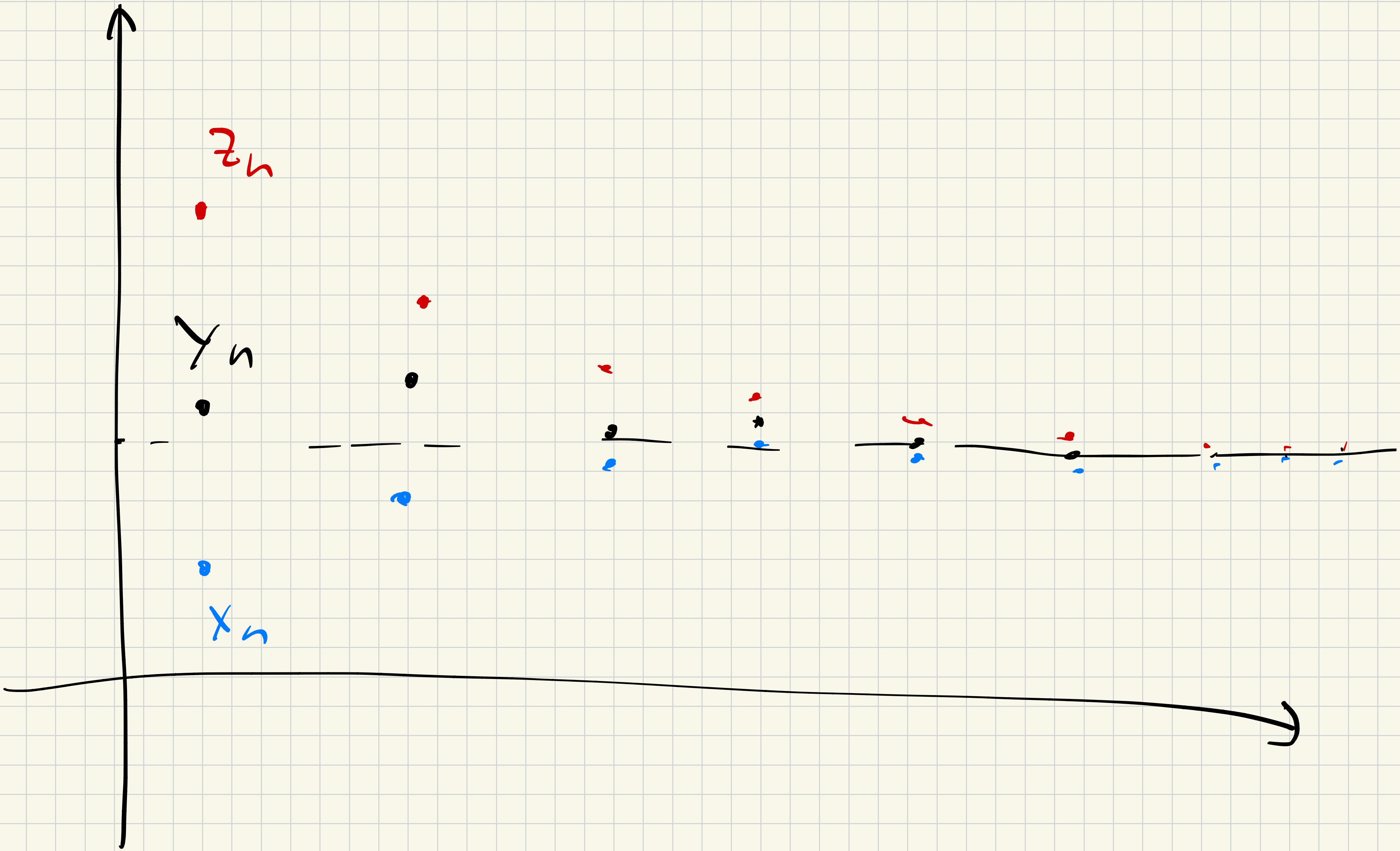
Analysis I Lecture 11

Last time:

Squeeze thm Let (x_n), (z_n) be two sequences s.t. $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = a$

Let (y_n) be such that $\exists N$ and $\forall n > N$ $x_n \leq y_n \leq z_n$ then

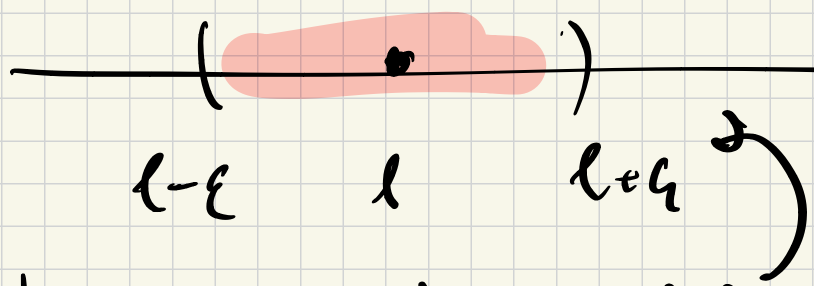
$$\lim_{n\to\infty} y_n = a.$$


z_n
y_n
x_n

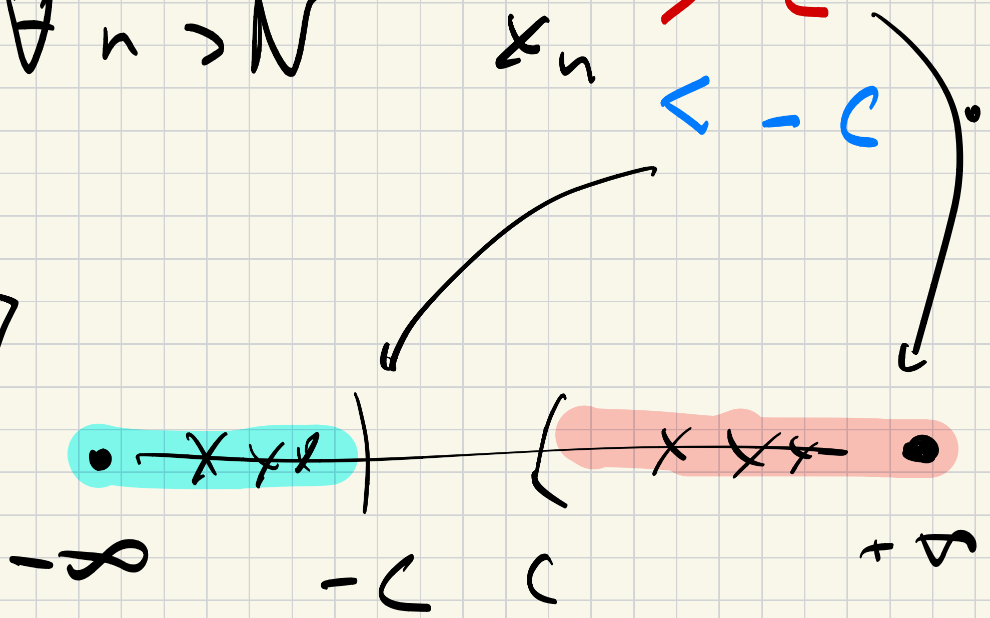
Sequences approaching infinity

We say that $\lim_{n\to\infty} x_n = \pm\infty$ if

$\forall\, C>0\ \exists\, N$ s.t. $\forall\, n>N$ $\quad x_n \begin{array}{l} > C \\ < -C \end{array}$



Usual limit def



Algebra of infinite limits

$$\lim_{n\to\infty} x_n = +\infty \qquad (y_n) \text{ bounded below}$$

then • $\lim_{n\to\infty} (x_n + y_n) = +\infty$

• If $\exists\, A > 0$ and $N \in \mathbb{N}$ with $y_n > A \ \forall\, n > N$

$$\lim_{n\to\infty} x_n \cdot y_n = +\infty$$

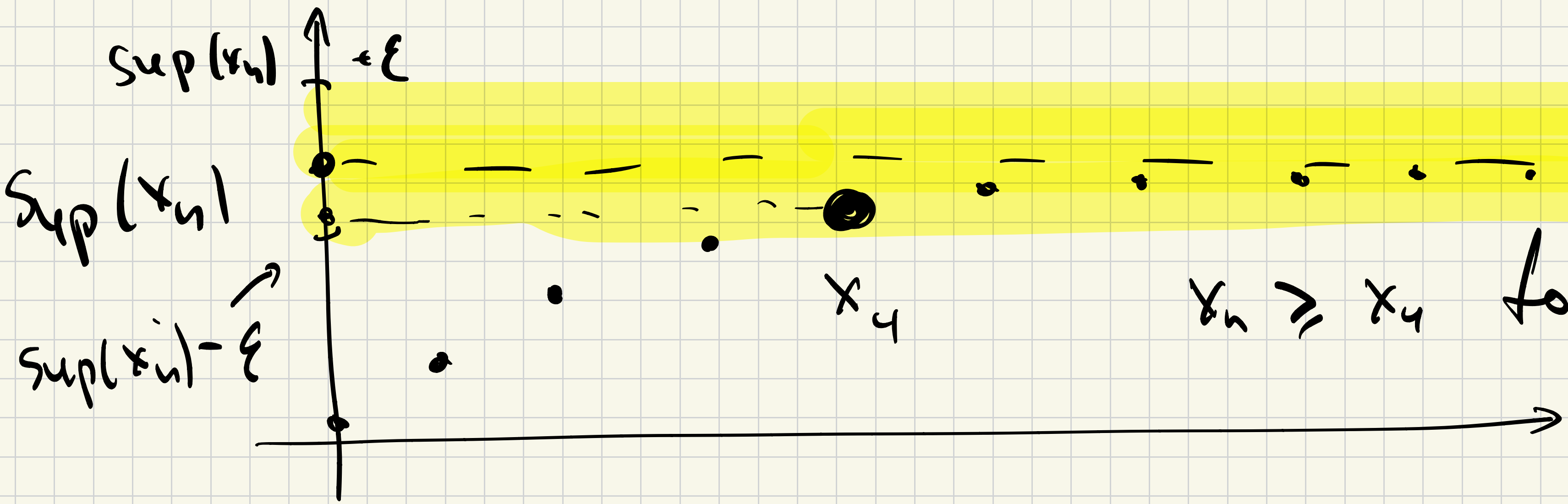
• If (y_n) is bounded then $\lim_{n\to\infty} \frac{y_n}{x_n} = 0$

Today:

- More convergence criteria
- Subsequences
- Liminf, Limsup

Theorem (Monotone convergence) $x_{n+1} \geq x_n$

Let (x_n) be monotonely increasing (decreasing) then $\lim_{n\to\infty} x_n = \sup(x_n)$ (or $\inf(x_n)$).



$x_n \geq x_4$ for $n \geq 4$

Theorem (d'Alamber criterion / Ratio test)

Let (x_n) be a sequence s.t. $x_n \neq 0 \ \forall n$

and s.t. $\lim\limits_{n\to\infty} \dfrac{|x_{n+1}|}{|x_n|} = \ell$ exists

then

if $\ell < 1$ then $\lim\limits_{n\to\infty} x_n = 0$

if $\ell > 1$ (x_n) diverges

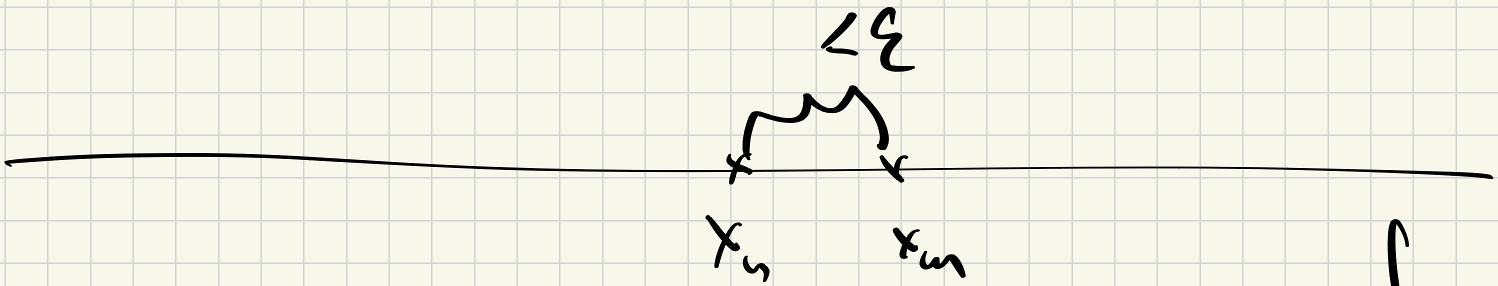
if $\ell = 1$ test is inconclusive

$x_n = (-1)^n$ diverges $\dfrac{|x_{n+1}|}{|x_n|} = 1$ | $x_n = c$ $\dfrac{|x_{n+1}|}{|x_n|} = 1$

Theorem (Cauchy criterion)

Let (x_n) be a sequence. Then (x_n) converges if and only if it is Cauchy sequence:

i.e. $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall n, m > N \quad |x_n - x_m| < \varepsilon$.



for $n, m > N$

How to use this?

Squize thm.: Some nice functions with inequalities

e.g. $0 \leq \sin\left(\frac{1}{n}\right) \leq \frac{1}{n}$ or $0 \leq 1 - \cos^2\left(\frac{1}{n}\right) \leq \frac{1}{n^2}$

Monotone convergence

- If no chance to describe limit.
- Sequences defined recursively.

d'Alembert (Ratio test)

Sequences for which it is easy to compute $\left|\frac{x_{n+1}}{x_n}\right|$

e.g α^n or $n!$

$$\left|\frac{x_{n+1}}{x_n}\right| = |\alpha| \qquad \frac{(n+1)!}{n!} = (n+1)$$

$x_{n+1} \geq x_n$ monotone ; $x_{n+1} > x_n$ strict monotone

Examples

1) Sequence $x_n = \left(1+\frac{1}{n}\right)^n$ is monotonely increasing and bounded hence it is converging

by geometric arithmetic mean inequality.

In fact $\lim_{n\to\infty} x_n = \sup(x_n)$.

Remark Number e is defined by $\lim_{n\to\infty} x_n = e$

2) $x_n = \frac{\alpha^n}{n!}$ Use d'Alambert:

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{\left|\frac{\alpha^{n+1}}{(n+1)!}\right|}{\left|\frac{\alpha^n}{n!}\right|} = \frac{n!}{(n+1)!} \cdot \frac{|\alpha|^{n+1}}{|\alpha|^n} =$$

$$= \frac{|\alpha|}{n+1} \xrightarrow[n\to\infty]{} 0 < 1$$

$\lim\limits_{n\to\infty} \frac{|\alpha|}{n+1} = 0$

$$\Rightarrow \lim_{n\to\infty} \frac{\alpha^n}{n!} = 0 \text{ by d'Alamber}$$

Subsequences

$$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \quad x_9 \quad x_{10} \ \ldots$$

$$\begin{array}{ccccc} x_2, & x_6, & x_7, & x_9 & \ldots \\ \| & \| & \| & \| & \\ y_0, & y_1, & y_2, & y_3, & \ldots \end{array}$$

Definition Let $(x_n)_{n\geq 0}$ be a sequence then a subsequence $(y_k)_{k\geq 0}$ of (x_n) is a sequence defined by $y_k = x_{n_k}$ for some strictly increasing function

$$f(k+1) > f(k)$$

$$\begin{aligned} f : \mathbb{N} &\longrightarrow \mathbb{N} \\ k &\longmapsto f(k) =: n_k \end{aligned}$$

$$y_k := x_{f(k)}$$

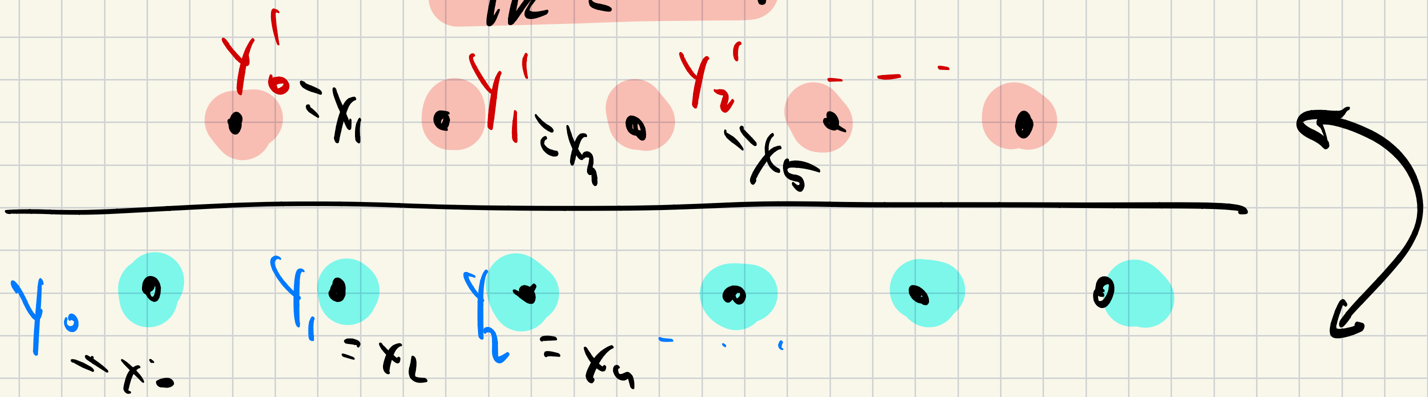
Example 1) $x_n = (-1)^n$

- $f(k) = 2k$: $\quad y_k = x_{f(k)} = x_{2k} = (-1)^{2k} = 1$

$$\boxed{y_k = 1}$$

- $f(k) = 2k+1$: $\quad y'_k = x_{f(k)} = x_{2k+1} = (-1)^{2k+1} = -1$

$$\boxed{y'_k = -1}$$



2) Let $\boxed{x_n = \left(1+\frac{2}{n}\right)^n}$ and let

$$f(k) = n_k = 2k$$

$$y_k = x_{2k} = \left(1+\frac{2}{2k}\right)^{2k} = \left(1+\frac{1}{k}\right)^{2k} =$$

$$= \underbrace{\left(1+\frac{1}{k}\right)^k}\cdot\underbrace{\left(1+\frac{1}{k}\right)^k}$$

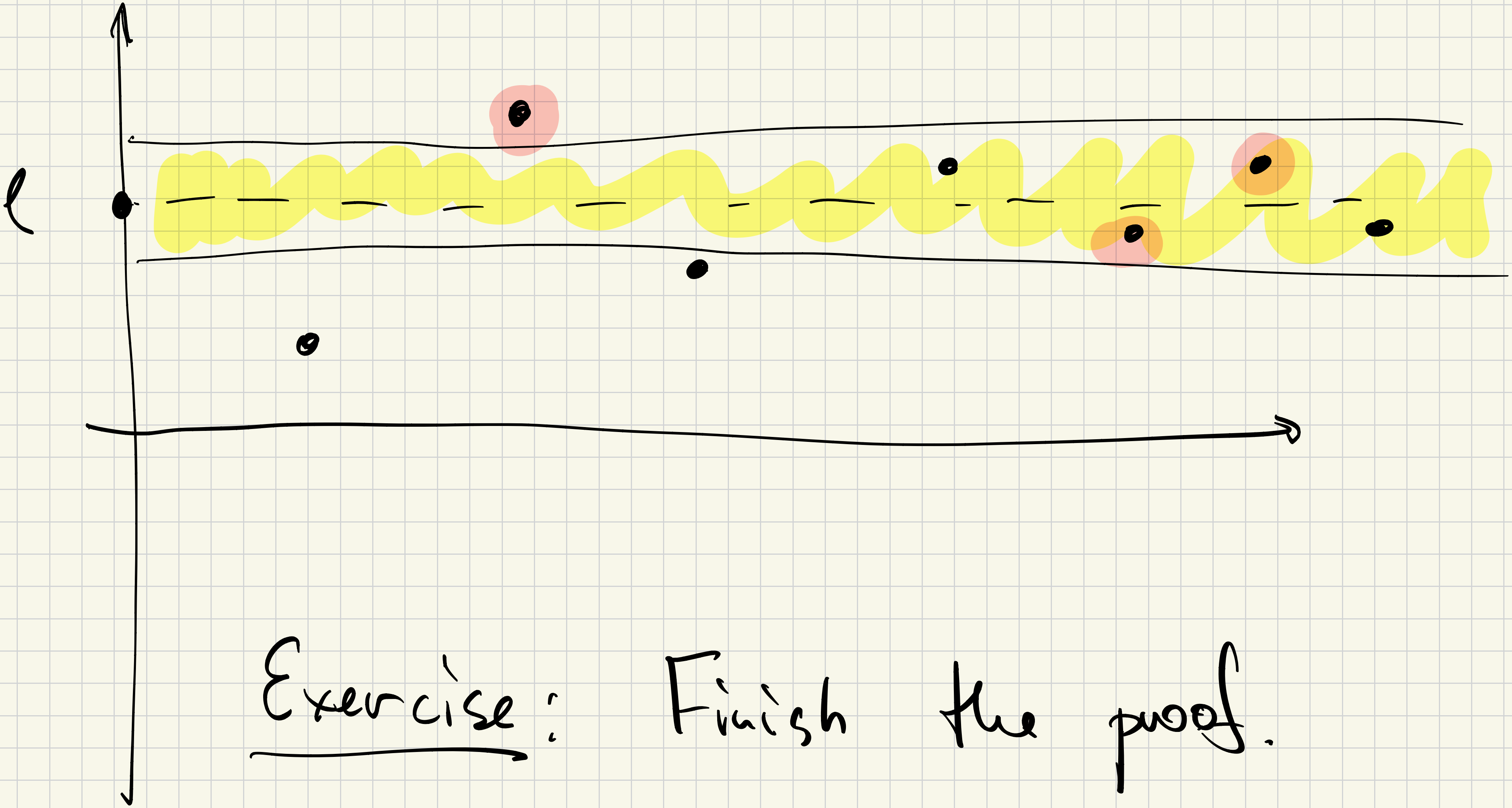
Remark

$$\lim_{k\to\infty} y_k = e^2$$

Convergence of subsequences

Proposition Let (x_n) be a sequence with $\lim_{n\to\infty} x_n = \ell$ then for every subsequence (y_k) of (x_n)

$$\lim_{k\to\infty} y_k = \ell .$$



Exercise: Finish the proof.

Definition An accumulation point of a sequence (x_n) is a limit of some subsequence.

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e.g. $x_n = (-1)^n$ we have two subsequences $y_k = 1$ $y'_k = -1$ with limitss 1 and -1. So ± 1 are accumulation points.

Theorem (Bolzano–Weierstrass)

Every bounded sequence admits a convergent subsequence.

In other words every bounded sequence has an accumulation point.

Easy examples

- $x_n = (-1)^n \rightsquigarrow$ $y_k = x_{2k}$, $y_k' = x_{2k+1}$ converge

- $x_n = \left|\sin\left(\frac{n\pi}{4}\right)\right| \cdot \left(1+\frac{1}{n}\right)^n$

periodic

converging

$$\sin\left(\frac{n\pi}{4}\right) = \begin{cases} 0 & \text{if } n = 4k \\ \frac{\sqrt{2}}{2} & \text{if } n = 4k+1 \text{ or } 4k+3 \\ 1 & \text{if } n = 4k+2 \end{cases}$$

So we can take

$$Y_k = X_{4k+0} = 0 \cdot \left(1+\frac{1}{n}\right)^n = 0$$

$$Y_k = X_{4k+1} = \frac{\sqrt{2}}{2} \cdot \left(1+\frac{1}{4k+1}\right)^{4k+1} \longrightarrow$$

$$\xrightarrow[n\to\infty]{} \frac{\sqrt{2}}{2} \cdot e$$

$$Y_k = X_{4k+2} = 1 \cdot \left(1+\frac{1}{4k+2}\right)^{4k+2} \longrightarrow e$$

$$k \to \infty$$

$\blacksquare$

Non-trivial example

$x_n = \sin(n)$ is a bounded sequence

But it not easy to give a convergent subsequence.